Exact Analytical Solutions of the Non Polynomial Oscillator

$$V(x) = x^2 + \frac{\lambda x^2}{(1+g x^2)}$$

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We derive new exact (analytical) solutions of the non polynomial oscillator $V(x) = x^2 + \lambda x^2 / (1 + g x^2)$ in one as well in three dimensions. The solutions are derived within the framework of supersymmetric quantum mechanics, and it is shown that exact solutions exist when the coupling constants satisfy a supersymmetric constraint.

Recently it has been shown [1] that the formalism of supersymmetric quantum mechanics (SUSYQM) can be profitably employed to determine exact analytical solutions of the Schrödinger equation and the corresponding eigenvalues. The method essentially depends on the construction of the superpotential so that the SUSY potential can be compared with the potential whose solution is saught. We shall find new solutions and energy eigenvalues for the non polynomial interaction (which has been studied extensively for exact as well as approximate solutions [2]) using a very simple algebraic procedure. It may be mentioned that in the case of shape invariant potentials the energy eigenvalues can be determined in a simple manner [3] using supersymmetry. However, the non polynomial interaction is not shape invariant and therefore exact solutions can not be found by the above method.

We recall that in one dimension a SYSYQM Hamiltonian consists of a pair of Hamiltonians [4]

$$H = \{Q^+, Q\} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \tag{1}$$

where Q^+, Q are the supercharges and are given by

$$Q^+ = (p - iW)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{2}$$

$$Q = (p + iW)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3}$$

$W(x)$ is called the superpotential and the choice of it is of crucial importance in deriving new solutions.

Using explicit forms of Q and Q^+ as given in (2) and (3) the pair of Hamiltonians in (1) can be written as

$$H_\pm = -\frac{d^2}{d x^2} + V_\pm(x), \tag{4}$$

$$V_\pm(x) = W^2(x) \pm W'(x). \tag{5}$$

The wave functions on which H operates are two component column vectors of the form

$$\varphi(x) = \begin{pmatrix} \varphi_+(x) \\ \varphi_-(x) \end{pmatrix}. \tag{6}$$

It may be pointed out that the ground state $|\Omega\rangle$ is always annihilated by the supercharges:

$$Q|\Omega\rangle = Q^+|\Omega\rangle = 0. \tag{7}$$

From (7) it follows that the ground state wave functions are of the form

$$\begin{pmatrix} \varphi_+^0(x) \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ \varphi_-^0(x) \end{pmatrix}, \tag{8}$$

where

$$\varphi_\pm^0(x) \sim \exp\left(\pm \int^x W(t)\, dt\right). \tag{8'}$$

We shall take the first wave function as physically acceptable if $\varphi_+^0(x)$ satisfies proper boundary conditions, and the latter if $\varphi_-^0(x)$ satisfies proper boundary conditions. But $\varphi_+^0(x)$ and $\varphi_-^0(x)$ can not satisfy proper boundary conditions at the same time, and of course it may happen that none of them are normalizable (in which case supersymmetry is spontaneously broken). If, however, either of $\varphi_\pm^0(x)$ is normalizable, supersymmetry is unbroken with the ground state energy equal to zero. It will turn out that the fact $E = 0$ for unbroken SUSY will play a key role in

determining the energy eigenvalues of the non polynomial oscillator.

Let us now turn to the choice of the superpotential. We make the following ansatz for the superpotential $W(x)$:

$$W(x) = x - \frac{c}{x} - \sum_{i=0}^{N} \frac{2 g_i x}{(1 + g_i x^2)} - \frac{4 h x^3}{(1 + h x^4)},$$

$$g_0 = 0, \quad g_1 = g. \tag{9}$$

We first consider the case $N = 1$. Our aim would be to identify the $(-)$ sector corresponding to (9) with the non polynomial oscillator potential

$$V(x) = x^2 + \frac{\lambda x^2}{(1 + g x^2)} = x^2 + \frac{\lambda}{g} - \frac{\lambda}{g} \frac{1}{(1 + g x^2)}, \tag{10}$$

and in three dimensions

$$V(x) = x^2 + \frac{\lambda}{g} - \frac{\lambda}{g} \frac{1}{(1 + g x^2)} + \frac{l(l+1)}{x^2}. \tag{11}$$

From (5) and (9) we have

$$V_-(x) = x^2 + \frac{c(c-1)}{x^2} - (2c + 13) \tag{12}$$

$$+ \frac{[8 h c + 12 h + 16 g_1/(1/g_1 + g_1/h)] x^2}{(1 + h x^4)}$$

$$+ \frac{[8 - 16/(1/g_1 + g_1/h)]}{(1 + h x^4)}$$

$$+ \frac{4 + 4 g_1 c + 18 g_1 - 16 g_1^2/h(1/g_1 + g_1/h)}{(1 + g_1 x^2)}.$$

Therefore, if we identify (12) with (10) and (11) we have

$$8 h c + 12 h + \frac{16 g_1}{(1/g_1 + g_1/h)} = 0, \tag{13}$$

$$8 = \frac{16}{(1/g_1 + g_1/h)}, \tag{14}$$

$$-\frac{\lambda}{g} = 4 + 4 g_1 c + 18 g_1 - \frac{16 g_1^2/h}{(1/g_1 + g_1/h)}, \tag{15}$$

$$c = 0, 1, l, +1, \tag{16}$$

and the energy eigenvalues are related through the relation

$$E - \lambda/g_1 = E_- + (2c + 13) \tag{17}$$

(where E denotes the energy corresponding to (10), (11), and E_- is the energy corresponding to $V_-(x)$). We note that for $W(x)$ given by (9) with $N = 1$,

$\exp(-\int^x W(t) \, dt)$ is normalizable and hence SUSY is unbroken (so that the ground state energy $E^0 = 0$).

To get the solutions and the corresponding eigenvalues it is now necessary to determine g_1 and h. From (13) and (14) we find

$$g_1 = \frac{2}{(2c + 7)}, \tag{18}$$

$$h = -\frac{4}{(2c + 3)(2c + 7)}, \tag{19}$$

and using these values the supersymmetric constraint is found to be

$$-\lambda = \frac{8}{(2c + 7)} \left[1 + \frac{2c + 9}{2c + 7} + \frac{2(2c + 3)}{(2c + 7)} \right]. \tag{20}$$

It is now easy to find the solutions and the corresponding energy eigenvalues given respectively by (8′) and (17). The solutions in one dimension can be found by putting $c = 0, 1$ and are given by

$$\varphi_-^0(x) \sim \left(1 + \frac{2}{7} x^2\right)\left(1 - \frac{4}{21} x^4\right) \exp\left(-\frac{x^2}{2}\right), \tag{21}$$

$$E = 3/7 \tag{22}$$

and

$$\varphi_-^0(x) \sim x \left(1 + \frac{2}{9} x^2\right)\left(1 - \frac{4}{45} x^4\right) \exp\left(-\frac{x^2}{2}\right), \tag{23}$$

$$E = 15/9. \tag{24}$$

The solution (23) is also valid in three dimensions with $l = 0$. The general solution in three dimension valid for any l is given by

$$\varphi_-^{0,l}(x) \sim x^{l+1}\left(1 + \frac{2}{21 + 9} x^2\right)\left(1 - \frac{4 x^4}{(21 + 5)(21 + 9)}\right)$$

$$\times \exp\left(-\frac{x^2}{2}\right), \tag{25}$$

$$E^l = 21 - \frac{12(21 + 7)}{(21 + 9)} + 11. \tag{26}$$

It is readily seen that the wave functions (21), (23) and (25) satisfy proper boundary conditions at $x = \pm \infty$ and $x = 0$. The set of solutions (21)–(26) constitutes new exact (analytical) solutions of the non polynomial oscillator.

We note that in the general case when $W(x)$ is given by (9) the energy eigenvalues are given by

$$E = \frac{\lambda}{g} + 2c + 9 + 4N, \tag{27}$$

$$\varphi^0_-(x) \sim x^c (1 + hx^4) \prod_{i=0}^{N} (1 + g_i x^2) \exp(-x^2/2), \tag{28}$$

where $c = 0, 1, l+1$ and the constants g_i's and h are determined from the following set of equations:

$$-\frac{\lambda}{g_1} = 4 + 4cg_1 + 18g_1 - \frac{16g_1^2}{1/g_1 + g_1/h}$$

$$+ \sum_{j=2}^{N} \frac{8g_1 g_j}{g_j - g_1}, \tag{29}$$

$$4 + 4cg_i + 18g_i - \frac{16g_i^2}{1/g_i + g_i/h} + \sum_{j \neq i}^{N} \frac{8g_i g_j}{g_j - g_i} = 0,$$

$$i = 2, \ldots, N, \tag{30}$$

$$8hc + 12h + \sum_{i=1}^{N} \frac{16g_i}{(1/g_i + g_i/h)} = 0. \tag{31}$$

A natural question that now arises is the following: How many magic terms are there which can be added to the superpotential to give exact solutions? As an answer to this question we present explicit forms of the superpotentials only and do not proceed to the determination of the exact solutions (since the proce-

dure in each case is a copy of the present one):

$$W(x) = x - \sum_{i=0}^{N} \frac{2g_i x}{(1 + g_i x^2)} - \sum_{j=2}^{M} \frac{2jh_j x^{2j-1}}{(1 + h_j x^{2j})}, \quad g_0 = 0, \tag{32}$$

where $h_j \neq 0$ for at least one value of j.

The most general form of the superpotential containing (possibly) all the magic terms is given by

$$W(x) = x - \frac{c}{x} - \sum_{i=1}^{N} \frac{2n_i g_i x^{2n_i-1}}{(1 + g_i x^{2n_i-1})}, \tag{33}$$

where n_i can assume any positive integral value and $g_i \neq 0$ for at least one value of i.

It has thus been shown that all the known exact solutions of the non polynomial oscillator as well as the new ones can be found from a unified approach within the framework of supersymmetric quantum mechanics. Therefore, a reasonable conjecture is that exact solutions of the non polynomial oscillator can exist only if the coupling constants satisfy a supersymmetric constraint.

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